ON TRANSPOSITIONAL RELATIONS IN MECHANICS

(O PERESTANOVOCHNYKH SOOTNOSHENIIAKH V MEKHANIKE)

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V. I. KIRGETOV
(Moscow)

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As soon as the nonholonomic relations, discovered by Hertz [1], were introduced into mechanics, there arose the question whether the already existing theorems for holonomic systems could be extended to nonholonomic ones. First the question was raised (by Hertz [1]) whether Hamilton's principle was valid for nonholonomic systems. Hertz [1] expressed doubt with respect to the validity of this principle for nonholonomic systems; Appell [3,2] asserted definitely that this principle need not hold for nonholonomic systems. If one examines the derivation of Hamilton's principle, it becomes clear that the validity of Hamilton's principle is based on the admissibility of transpositional relations

$$d\delta x = \delta dx, \qquad d\delta y = \delta dy, \qquad d\delta z = \delta dz$$

for all coordinates of the system.

Kirchhoff [4] proved that these relations hold for holonomic systems, Appell [2] shows by an example that for a nonholonomic system the relations mentioned may fail to hold for all coordinates. Subsequently the question of the validity of these relations has had the attention of various scientists. Hamel [5], for example, considered it. Finally, in recent years, this topic has been treated in a number of works by Soviet scholars.

1. Let a material system be given. Let $q_1, \ldots, q_k$ be generalized coordinates of the system. We suppose that the system is subjected to linear differential constraints of the type

$$q_\vartheta' + \sum_{\tau=1}^{k-m} a_{\vartheta,m+\tau}\, q_{m+\tau}' + a_\vartheta = 0 \qquad (\vartheta = 1, \ldots, m) \tag{1.1}$$

with coefficients that are differentiable in some region A.

The motion

$$q_\nu = \varphi_\nu(t) \qquad (\nu = 1, \ldots, k) \tag{1.2}$$

of the material system is said to be kinematically admissible if the functions $\varphi_\nu(t)$ satisfy identically the system of equations (1.1).

The following theorem is true for a sufficiently small neighborhood of every point $q_1^0, \ldots, q_k^0, t^0$ in the region A.

Every kinematically admissible motion (1.2) of the given material system, which satisfies the condition $\varphi_\nu(t^0) = q_\nu^0 (\nu = 1, \ldots, k)$, can be given by a one-parameter family of kinematically admissible motions

$$q_\nu = \Phi_\nu(t, \alpha) \qquad (\nu = 1, \ldots, k) \tag{1.3}$$

where α is an arbitrary parameter such that the functions $\Phi_\nu(t, \alpha)$ satisfy the identities

$$\Phi_\nu(t, 0) \equiv \varphi_\nu(t), \qquad \frac{\partial^2 \Phi_\nu}{\partial\alpha\partial t} \equiv \frac{\partial^2 \Phi_\nu}{\partial t\partial\alpha} \tag{1.4}$$

where $k - m$ of these functions can be chosen arbitrarily except for the conditions imposed by (1.4).

Proof. Let us select twice-differentiable functions $\Phi_{m+1}, \ldots, \Phi_k$ satisfying the conditions (1.4), but otherwise being arbitrary. We substitute these functions in the equations of the system (1.1). That system can then be written in the form

$$q_\vartheta' = f_\vartheta(t, \alpha, q_1, \ldots, q_m) \qquad (\vartheta = 1, \ldots, m) \tag{1.5}$$

The right-hand members of the equations of this system will be differentiable functions of all indicated variables. This system has only one solution $q_\theta = \varphi_\theta(t, \alpha)(\theta = 1, \ldots, m)$, satisfying the conditions $\Phi_\theta(t^0, \alpha) = q_\theta^0 \ (\theta = 1, \ldots, m)$.

The system of functions

$$\Phi_1(t, \alpha), \ldots, \Phi_m(t, \alpha), \quad \Phi_{m+1}(t, \alpha), \ldots, \Phi_k(t, \alpha) \tag{1.6}$$

defines a one-parameter family of kinematically admissible motions of the system. We shall show that this family of motions satisfies the conditions (1.4).

Indeed, from the fact that for $q_{m+\sigma} = \varphi_{m+\sigma}(t) \ (\sigma = 1, \ldots, k - m)$ the system (1.1) has a unique solution passing through the point $q_1^0, \ldots, q_k^0, t^0$, namely, $q_\theta = \varphi_\theta(t)(\theta = 1, \ldots, m)$, and from the fact that $\Phi_{m+\sigma}(t, 0) \equiv \varphi_{m+\sigma}(t) \ (\sigma = 1, \ldots, k - m)$, it follows that $\Phi_\theta(t, 0) = \varphi_\theta(t) (\theta = 1, \ldots, m)$.

Thus, the system of functions (1.6) satisfies the first of the identities (1.4). We now show that it also satisfies the second identity (1.4).

For this purpose we substitute the functions $\Phi_1(t, a), \ldots, \Phi_m(t, a)$ in the equations of the system (1.5), and obtain

$$\frac{\partial\Phi_\vartheta(t,\alpha)}{\partial t} \equiv F_\vartheta(t,\alpha) \qquad (\vartheta = 1, \ldots, m) \tag{1.7}$$

The functions $F_\theta(t, a)(\theta = 1, 2, \ldots, m)$, obviously, will be differentiable functions of t and a. Differentiating the members of the identity (1.7) with respect to a, we obtain

$$\frac{\partial^2\Phi_\vartheta(t,\alpha)}{\partial\alpha\partial t} = \frac{\partial F_\vartheta(t,\alpha)}{\partial\alpha} \qquad (\vartheta = 1, \ldots, m) \tag{1.8}$$

On the other hand, one may write the system of identities (1.7) in the form

$$\Phi_\vartheta(t,\alpha) \equiv \int_{t^\circ}^{t} F_\vartheta(t,\alpha)\,dt + q_\vartheta^\circ \qquad (\vartheta = 1, \ldots, m)$$

Differentiating these expressions with respect to a, and performing the differentiation under the integral sign (which is permissible owing to the fact that the functions $F_\theta(t, a)(\theta = 1, \ldots, m)$ are assumed to have continuous derivatives of all orders with respect to a), and differentiating the result with respect to t, we finally obtain

$$\frac{\partial^2\Phi_\vartheta(t,\alpha)}{\partial t\partial\alpha} \equiv \frac{\partial F_\vartheta(t,\alpha)}{\partial\alpha} \qquad (\vartheta = 1, \ldots, m)$$

Combining these identities with those given in (1.8), we obtain the second identity of (1.4). This establishes the lemma.

Let an arbitrary kinematically admissible motion of a material system be given by the equations

$$q_\nu = \varphi_\nu(t) \qquad (\nu = 1, \ldots, k) \tag{1.9}$$

with the initial conditions $\phi_\nu(t^\circ) = q_\nu^\circ(\nu = 1, \ldots, k)$. We include it in the one-parameter family

$$q_\nu = \Phi_\nu(t,\alpha) \qquad (\nu = 1, \ldots, k) \tag{1.10}$$

of kinematically admissible motions of a material system satisfying (1.7).

The system of quantities

$$\delta q_\nu = \alpha\left[\frac{\partial\Phi_\nu(t,\alpha)}{\partial\alpha}\right]_{\alpha=0} \qquad (\nu = 1, \ldots, k)$$

is called the variation of the motion (1.9) subjected to inclusion within the family (1.10) of kinematically admissible motions of the system.

The quantities $\delta q_1, \ldots, \delta q_k$ are, obviously, differentiable functions of time.

We call attention to the fact that the quantities δq_{m+1}, δq_{m+2}, ..., δq_k may be chosen arbitrarily. This is a simple consequence of the lemma.

Next we prove the following theorem.

If the variations of every kinematically admissible motion, subjected to inclusion in all possible families of kinematically admissible motions of the system, satisfy all possible transpositional relations, then the system of equations of constraints (1.1) is completely integrable.

Proof. In the neighborhood of an arbitrary point q_1°, ..., q_k°, t° of the region A, we select an arbitrary kinematically admissible motion $q_\nu = \phi_\nu(t)(\nu = 1, \ldots, k)$ satisfying the conditions $\phi_\nu(t^\circ) = q_\nu^\circ(\nu = 1, \ldots, k)$. We include this motion in a one-parameter family $q_\nu = \Phi_\nu(t, a)$ $(\nu = 1, \ldots, k)$ of kinematically admissible motions satisfying the conditions (1.4), which can be done in consequence of the established lemma.

The equalities

$$\frac{\partial}{\partial t}\Phi_\vartheta + \sum_{\sigma=1}^{k-m} a_{\vartheta,m+\sigma}^{(1)} \frac{\partial}{\partial t}\Phi_{m+\sigma} + a_\vartheta^{(1)} = 0 \qquad (\vartheta = 1, 2, \ldots, m)$$

where

$$a_{\vartheta,m+\sigma}^{(1)} = a_{\vartheta,m+\sigma}(t, \Phi_1(t, \alpha), \ldots, \Phi_k(t, \alpha))$$

$$a_\vartheta^{(1)} = a_\vartheta(t, \Phi_1(t, \alpha), \ldots, \Phi_k(t, \alpha))$$

must hold identically in t and α. It follows from these equalities and from (1.4) that

$$\frac{d}{dt}\delta q_\vartheta + \sum_{\sigma=1}^{k-m} a_{\vartheta,m+\sigma}^{*} \frac{d}{dt}\delta q_{m+\sigma} + \sum_{\sigma=1}^{k-m} \frac{d}{dt}\varphi_{m+\sigma}\delta a_{\vartheta,m+\sigma} + \delta a_\vartheta = 0 \quad (\vartheta = 1, \ldots, m) \tag{1.11}$$

Here we have introduced the notation

$$\delta a_{\vartheta,m+\sigma} = \sum_{\nu=1}^{k}\left(\frac{\partial a_{\vartheta,m+\sigma}}{\partial q_\nu}\right)^{*}\delta q_\nu, \qquad \delta a_\vartheta = \sum_{\nu=1}^{k}\left(\frac{\partial a_\vartheta}{\partial q_\nu}\right)^{*}\delta q_\nu \qquad \begin{pmatrix}\vartheta = 1, \ldots, m \\ \sigma = 1, \ldots, k-m\end{pmatrix}$$

In this section the asterisk will indicate expressions in which the coordinates q_1, ..., q_k are replaced by the corresponding functions $\phi_1(t)$, ..., $\phi_k(t)$.

By the hypotheses of the theorem, the quantities δq_1, ..., δq_k must satisfy identically all transpositional relations. In other words, we have the identities

$$\delta q_\vartheta + \sum_{\sigma=1}^{k-m} a_{\vartheta,m+\sigma}^{*}\delta q_{m+\sigma} = 0 \qquad (\vartheta = 1, \ldots, m) \tag{1.12}$$

Differentiating them with respect to time, we obtain

$$\frac{d}{dt}\delta q_\vartheta + \sum_{\sigma=1}^{k-m} a^{\bullet}_{\vartheta,m+\sigma}\frac{d}{dt}\delta q_{m+\sigma} + \sum_{\sigma=1}^{k-m}\delta q_{m+\sigma}\frac{d}{dt}a^{\bullet}_{\vartheta,m+\sigma} = 0 \qquad (\vartheta = 1,\ldots,m)$$

Let us subtract from the members of these equations the corresponding members of the equations (1.11) and obtain the system of equations

$$\sum_{\sigma=1}^{k-m}\delta q_{m+\sigma}\frac{d}{dt}a^{\bullet}_{\vartheta,m+\sigma} - \sum_{\sigma=1}^{k-m}\delta a_{\vartheta,m+\sigma}\frac{d}{dt}\varphi_{m+\sigma} - \delta a_\vartheta = 0 \qquad (\vartheta = 1,\ldots,m)$$

Making use of the expression (1.1) and (1.12) we eliminate from the last displayed system of equations the velocities $\phi'_1, \ldots, \phi'_m$ and the quantities $\delta q_1, \ldots, \delta q_m$. These equations are thus reduced to the form

$$\sum_{\tau=0}^{k-m}\delta q_{m+\tau}\Big[\Big(\frac{\partial a_{\vartheta,m+\tau}}{\partial t} - \sum_{\rho=1}^{m} a_\rho\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\rho} - \frac{\partial a_\vartheta}{\partial q_{m+\tau}} + \sum_{\rho=1}^{m} a_{\rho,m+\tau}\frac{\partial a_\vartheta}{\partial q_\rho}\Big) +$$
$$+\sum_{\sigma=1}^{k-m}\varphi'_{m+\sigma}\Big(\frac{\partial a_{\vartheta,m+\tau}}{\partial q_{m+\sigma}} - \sum_{\rho=1}^{m} a_{\rho,m+\sigma}\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\rho} - \frac{\partial a_{\vartheta,m+\sigma}}{\partial q_{m+\tau}} +$$
$$+\sum_{\rho=1}^{m} a_{\rho,m+\tau}\frac{\partial a_{\vartheta,m+\sigma}}{\partial q_\rho}\Big)\Big]^{\bullet} = 0 \qquad (\vartheta = 1,\ldots,m)$$

These identities must hold for arbitrary $\delta q_{m+1}, \ldots, \delta q_k$ under a given motion $q_\nu = \phi_\nu(t)$ $(\nu = 1, \ldots, k)$, and for arbitrary $\phi'_{m+1}, \phi'_{m+2}, \ldots, \phi'_k$ at the given point $q_1, \ldots, q_k^0$, t^0. The coefficients of $\delta q_{m+1}, \ldots, \delta q_k$ and of $\phi'_{m+1}, \phi'_{m+2}, \ldots, \phi'_k$ must, therefore, be zero in the above identities. But since the point $q_1^0, \ldots, q_k^0$, t^0 was chosen arbitrarily in the region A, it follows that the following set of identities

$$\frac{\partial a_{\vartheta,m+\tau}}{\partial t} - \sum_{\rho=1}^{m} a_\rho\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\rho} - \frac{\partial a_\vartheta}{\partial q_{m+\tau}} + \sum_{\rho=1}^{m} a_{\rho,m+\tau}\frac{\partial a_\vartheta}{\partial q_\rho} = 0$$

$$\frac{\partial a_{\vartheta,m+\tau}}{\partial q_{m+\sigma}} - \sum_{\rho=1}^{m} a_{\rho,m+\sigma}\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\rho} - \frac{\partial a_{\vartheta,m+\sigma}}{\partial q_{m+\tau}} + \sum_{\rho=1}^{m} a_{\rho,m+\tau}\frac{\partial a_{\vartheta,m+\sigma}}{\partial q_\rho} = 0$$
$$(\sigma, \tau = 1, \ldots, k-m)$$

must hold in the entire region A. This means that the system of equations (1.1) is completely integrable, as was to be proven.

Thus, in the case of nonholonomic systems, we cannot interpret all possible transpositions of the system as variations of the motion of the system within the class of all kinematically admissible motions of the system.

2. Let us suppose, as before, that a given material system is subjected to the differential constraints (1.1).

1. First, we make the following two hypotheses.

In the neighborhood of an arbitrary point $q_1^0, \ldots, q_k^0$ of the region A, and at a fixed time t^0, it is possible to select (with a given amount of arbitrariness) a continuously differentiable field of velocities satisfying (1.1) at the given time.

The field of velocities is given by the system of equations

$$q_\nu' = \psi_\nu(q_1, \ldots, q_k) \qquad (\nu = 1, \ldots, k)$$

$$\psi_\vartheta = -\sum_{\sigma=1}^{k-m} a_{\vartheta, m+\sigma}^{(2)} \psi_{m+\sigma} - a_\vartheta^{(2)} \qquad (\vartheta = 1, \ldots, m) \tag{2.1}$$

$$a_{\vartheta, m+\sigma}^{(2)} = a_{\vartheta, m+\sigma}(t^\circ, q_1, \ldots, q_k), \qquad a_\vartheta^{(2)} = a_\vartheta(t^\circ, q_1, \ldots, q_k)$$

where the functions $\phi_{m+1}, \ldots, \phi_k$ are continuously differentiable in the indicated arguments but are otherwise arbitrary.

Let
$$q_\nu = \varphi_\nu(t) \qquad (\nu = 1, \ldots, k) \tag{2.2}$$

be any kinematically admissible motion of a material system satisfying the conditions

$$\varphi_\nu(t^\circ) = q_\nu^\circ \qquad (\nu = 1, \ldots . k,) \tag{2.3}$$

At any instant of time one can select a possible displacement of the system so that in the process of motion there is obtained some chain of possible displacements. The second one of our initial hypotheses asserts the following: in the neighborhood of an arbitrary point $q_1^0, \ldots, q_k^0$, t^0 of the region A, it is possible to select, with a known degree of arbitrariness, a continuously differentiable (with respect to time) chain of possible displacements of the system.

Temporarily denoting the possible displacements of the system by the sequence $\pi q_1, \ldots, \pi q_k$, we may write the resulting chain of displacements as

$$\pi q_\nu = \eta_\nu(t) \qquad (\nu = 1, \ldots, k)$$

$$\eta_\vartheta(t) = -\sum_{\tau=1}^{k-m} a_{\vartheta, m+\tau}^{(1)} \eta_{m\ \tau}(t) \qquad (\vartheta = 1, \ldots, m) \tag{2.4}$$

$$a_{\vartheta, m+\tau}^{(1)} = a_{\vartheta, m+\tau}(t, \varphi_1, (t), \ldots, \varphi_k(t))$$

where the functions $\eta_{m+1}(t), \ldots, \eta_k(t)$ are continuously differentiable but otherwise arbitrary.

2. We introduce an operation δ which has the following properties:

(a) this operation is applicable only to functions of coordinates and time which are differentiable with respect to time;

(b) the result of the application of this δ-operation to the function $f(t, q_1, \ldots, q_k)$ at the point $q_1^0, \ldots, q_k^0, t^0$ is written as

$$\sum_{\nu=1}^{k} \left(\frac{\partial f}{\partial q_\nu}\right)_0 \delta q_\nu$$

where the partial derivatives are taken at the point $q_1^0, \ldots, q_k^0, t^0$, while the set of quantities $\delta q_1, \ldots, \delta q_k$ represents an arbitrary displacement of the system at the considered instant of time.

3. Let us take an arbitrary point $q_1^0, \ldots, q_k^0, t^0$ in the region A, and select within the neighborhood of this point at the fixed time t^0 a continuously differentiable field of velocities. This field is determined by the formulas (2.1). Applying the operation δ to the equations (2.1) (which are here considered as identities) we obtain the following equations which hold at the point $q_1^0, \ldots, q_k^0, t^0$:

$$\delta q_\mu' = \sum_{\nu=1}^{k} \left(\frac{\partial \psi_\mu}{\partial q_\nu}\right)_0 \delta q_\nu \qquad \begin{pmatrix}\mu = 1, \ldots, k\\ \nu = 1, \ldots, k\\ \vartheta = 1, \ldots, m\end{pmatrix}$$

$$\left(\frac{\partial \psi_\vartheta}{\partial q_\nu}\right)_0 = -\sum_{\sigma=1}^{k-m} (a_{\vartheta,m+\sigma})_0 \left(\frac{\partial \psi_{m+\sigma}}{\partial q_\nu}\right)_0 - \sum_{\sigma=1}^{k-m} (\psi_{m+\sigma})_0 \left(\frac{\partial a_{\vartheta,m+\sigma}}{\partial q_\nu}\right)_0 - \left(\frac{\partial a_\vartheta}{\partial q_\nu}\right)_0$$

Introducing the notation

$$\left(\frac{\partial \psi_\mu}{\partial q_\nu}\right)_0 = (\alpha_{\mu,\nu})_0$$

and then dropping the subscript zero, since the point $q_1^0, \ldots, q_k^0, t^0$ was an arbitrary point of the region A, we may write

$$\delta q_\mu' = \sum_{\nu=1}^{k} \alpha_{\mu,\nu} \delta q_\nu \quad (\mu = 1, \ldots, k)$$

$$\alpha_{\vartheta,\nu} = -\sum_{\sigma=1}^{k-m} a_{\vartheta,m+\sigma} \alpha_{m+\sigma,\nu} - \sum_{\sigma=1}^{k-m} q'_{m+\sigma} \frac{\partial a_{\vartheta,m+\sigma}}{\partial q_\nu} - \frac{\partial a_\vartheta}{\partial q_\nu}$$

Eliminating the coefficient $a_{\theta\nu}$ in these equations we finally obtain

$$\delta q_\vartheta' = -\sum_{\sigma=1}^{k-m} a_{\vartheta,m+\sigma} \sum_{\nu=1}^{k} \alpha_{m+\sigma,\nu} \delta q_\nu - \sum_{\sigma=m}^{k-m} q'_{m+\sigma} \delta a_{\vartheta,m+\sigma} - \delta a_\vartheta \qquad \begin{pmatrix}\vartheta = 1, \ldots, m\\ \sigma = 1, \ldots, k-m\end{pmatrix}$$

$$\delta q'_{m+\sigma} = \sum_{\nu=1}^{k} \alpha_{m+\sigma,\nu} \delta q_\nu \tag{2.5}$$

Thus, the sequence $\delta q_1, \ldots, \delta q_k$ is completely determined if our hypotheses are satisfied, and if we are given the generalized velocities of the system, its possible displacements, and the coefficient matrix

$$\left\| \begin{matrix} \alpha_{m+1,1} & \alpha_{m+1,2} \ldots \alpha_{m+1,k} \\ \alpha_{m+2,1} & \alpha_{m+2,2} \ldots \alpha_{m+2,k} \\ \ldots & \ldots \\ \alpha_{k,1} & \alpha_{k,2} \ldots \alpha_{k,k} \end{matrix} \right\| \tag{2.6}$$

4. In the neighborhood of an arbitrary point $q_1^0, \ldots, q_k^0, t^0$ of the region A, let us take a kinematically admissible motion (2.2) satisfying the conditions (2.3). Along the path of this motion we select a continuously differentiable chain of possible displacements of the system. This chain is determined by the equations (2.4) in which, however, the terms $\pi q_1, \ldots, \pi q_k$ have to be replaced by $\delta q_1, \ldots, \delta q_k$. Differentiating the modified equations (2.4) and setting $t = t^0$ in them, we obtain the following set of equations which must hold at the point $q_1^0, \ldots, q_k^0, t^0$

$$\begin{gathered} \frac{d}{dt}\delta q_\nu = \left(\frac{d\eta_\nu}{dt}\right)_0 \\ \left(\frac{d\eta_\vartheta}{dt}\right)_0 = -\sum_{\tau=1}^{k-m} (a_{\vartheta,m+\tau})_0 \left(\frac{d\eta_{m+\tau}}{dt}\right)_0 - \sum_{\tau=1}^{k-m} (\eta_{m+\tau})_0 (b_{\vartheta,m+\tau})_0 \\ (b_{\vartheta,m+\tau})_0 = \left(\frac{\partial a_{\vartheta,m+\tau}}{\partial t}\right)_0 + \sum_{\nu=1}^{k} \left(\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\nu}\right)_0 \left(\frac{d\varphi_\nu}{dt}\right)_0 \\ (\nu = 1, \ldots, k, \quad \vartheta = 1, \ldots, m, \quad \tau = 1, 2, \ldots, k-m) \end{gathered} \tag{2.7}$$

We should note that the following equalities hold:

$$(\eta_{m+\tau})_0 = (\delta q_{m+\tau})_0 \qquad (\tau = 1, \ldots, k-m) \qquad \left(\frac{d\varphi_\nu}{dt}\right)_0 = (q_\nu')_0 \qquad (\nu = 1, \ldots, k), \tag{2.8}$$

This means that the time derivative of the chain of possible displacements is determined at the point $q_1^0, \ldots, q_k^0, t^0$ as soon as there are given the generalized velocities of the system at this point, the possible displacements, and the set of quantities

$$\left(\frac{d\eta_{m+1}}{dt}\right)_0, \quad \left(\frac{d\eta_{m+2}}{dt}\right)_0, \ldots, \left(\frac{d\eta_k}{dt}\right)_0$$

This last set of quantities can be represented in the form

$$\left(\frac{d\eta_{m+\tau}}{dt}\right)_0 = (\beta_{m+\tau})_0 + \sum_{\nu=1}^{k} (\beta_{m+\tau,\nu})_0 (\delta q_\nu)_0 \qquad (\tau = 1, \ldots, k-m)$$

Let us suppose, however, that only chains of possible displacements are taken into consideration for which the quantities $(\eta'_{m+2})_0$, $(\eta'_{m+2})_0, \ldots, (\eta'_k)_0$ all vanish whenever the possible displacements $(\delta q_1)_0, \ldots, (\delta q_k)_0$ vanish. One must then set $(\beta_{m+1})_0 = \ldots = (\beta_k)_0 = 0$, which implies that

$$\left(\frac{d\eta_{m+\tau}}{dt}\right)_0 = \sum_{\nu=1}^{k} (\beta_{m+\tau,\nu})_0 (\delta q_\nu)_0 \qquad (\tau = 1, \ldots, k-m) \tag{2.9}$$

Because of the arbitrariness of the point $q_1^0, \ldots, q_k^0, t^0$, one may drop the subscript zero in the equations (2.7), (2.8), and (2.9). Eliminating the quantities $\eta_1', \ldots, \eta_m'$ from equation (2.7) and making use of (2.8) and (2.9) we can reduce (2.7) to the form

$$\frac{d}{dt}\delta q_{m+\tau} = \sum_{\nu=1}^{k} \beta_{m+\tau,\nu}\delta q_\nu$$

$$\frac{d}{dt}\delta q_\vartheta = -\sum_{\tau=1}^{k-m} a_{\vartheta,m+\tau}\sum_{\nu=1}^{k}\beta_{m+\tau,\nu}\delta q_\nu - \sum_{\tau=1}^{k-m} b_{\vartheta,m+\tau}\delta q_{m+\tau} \tag{2.10}$$

$$b_{\vartheta,m+\tau} = \frac{\partial a_{\vartheta,m+\tau}}{\partial t} + \sum_{\nu=1}^{k}\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\nu} q_\nu' \qquad \begin{pmatrix}\vartheta = 1, \ldots, m \\ \tau = 1, \ldots, k-m\end{pmatrix}$$

Thus, the time derivative of the chain of possible displacements for a material system, which at the time t^0 has the position $q_1^0, \ldots, q_k^0$, is determined as soon as there are given the generalized velocities of the system, its possible displacements, and the coefficient matrix

$$\begin{Vmatrix} \beta_{m+1,1} & \beta_{m+1,2} \cdots \beta_{m+1,k} \\ \beta_{m+2,1} & \beta_{m+2,2} \cdots \beta_{m+2,k} \\ \cdots & \cdots \\ \beta_{k,1} & \beta_{k,2} \cdots \beta_{k,k} \end{Vmatrix} \tag{2.11}$$

5. Equating the quantities $\delta q_\nu'$ to the corresponding (with respect to subscript ν) quantities $d\delta q_\nu/dt$ for the same values of the defining parameters, we derive, by means of (2.5) and (2.10), the following system:

$$\delta q_{m+\tau}' = \frac{d}{dt}\delta q_{m+\tau}$$

$$\delta q_\vartheta' - \frac{d}{dt}\delta q_\vartheta = \sum_{\tau=1}^{k-m} b_{\vartheta,m+\tau}\delta q_{m+\tau} - \sum_{\tau=1}^{k-m} q_{m+\tau}'\delta a_{\vartheta,m+\tau} - \delta a_\vartheta \tag{2.12}$$

$$b_{\vartheta,m+\tau} = \frac{\partial a_{\vartheta,m+\tau}}{\partial t} + \sum_{\nu=1}^{k}\frac{\partial a_{\vartheta,m+\tau}}{\partial q_\nu} q_\nu' \qquad \begin{pmatrix}\vartheta = 1, \ldots, k \\ \tau = 1, \ldots, k-m\end{pmatrix}$$

On this basis we conclude that the operations δ and $d(\ )/dt$ are commutative (transpositional) for the last $k - m$ generalized coordinates; this transposition is valid at every point of the region A independently of the values of the generalized velocities $q'_{m+1}, \ldots, q'_k$ and possible displacement. The situation in regard to analogous transpositional relations for the remaining coordinates is described by the theorem.

In order that the δ-operation at a given point shall satisfy the transpositional relations

$$\delta q'_{s+i} = \frac{d}{dt}\delta q_{s+i} \qquad (i = 1, \ldots, m - s) \tag{2.13}$$

it is necessary and sufficient that the following equations hold at that point

$$\frac{\partial a_{s+i,m+\sigma}}{\partial q_{m+\tau}} + \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\tau}}{\partial q_\rho} a_{\rho,m+\sigma} - \frac{\partial a_{s+i,\,m+\tau}}{\partial q_{m+\sigma}} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_{\rho,m+\tau} = 0$$

$$\frac{\partial a_{s+i,\,m+\sigma}}{\partial t} + \sum_{\rho=1}^{m} \frac{\partial a_{s+i}}{\partial q_\rho} a_{\rho,m+\sigma} - \frac{\partial a_{s+i}}{\partial q_{m+\sigma}} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_\rho = 0 \tag{2.14}$$

$$(\sigma = 1, \ldots, k - m;\ \tau = 1, \ldots, k - m;\ i = 1, \ldots, m - s)$$

Proof of Necessity. Taking into consideration the relations (2.13) we obtain from (2.12) the equations

$$\sum_{\sigma=1}^{k-m} b_{s+i,\,m+\sigma} q_{m+\sigma\delta} - \sum_{\tau=1}^{k-m} q'_{m+\tau} \delta a_{s+i,\,m+\tau} - \delta a_{s+i} = 0 \quad (\sigma = 1, \ldots, k - m)$$

$$b_{s+i,\,m+\sigma} = \frac{\partial a_{s+i,\,m+\sigma}}{\partial t} + \sum_{\nu=1}^{k} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\nu} q_\nu' \qquad (i = 1, \ldots, m - s) \tag{2.15}$$

By isolating within these equations the expression $\delta a_{s+i,\,m+\tau}$ and δa_{s+i}, and by then eliminating the quantities $q_1, \ldots, q_m;\ q_1', \ldots, q_m'$ with the aid of the relations

$$q_\vartheta' + \sum_{\tau=1}^{k-m} a_{\vartheta,m+\tau} q'_{m+\tau} + a_\vartheta = 0, \quad \delta q_\vartheta + \sum_{\tau=1}^{k-m} a_{\vartheta,m+\tau} \delta q_{m+\tau} = 0 \quad (\vartheta = 1, \ldots, m)$$

we reduce the identities (2.15) to the form

$$\sum_{\sigma=1}^{k-m} \delta q_{m+\sigma} \Bigg[\Bigg(\frac{\partial a_{s+i,\,m+\sigma}}{\partial t} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_\rho - \frac{\partial a_{s+i}}{\partial q_{m+\sigma}} + \sum_{\rho=1}^{m} \frac{\partial a_{s+i}}{\partial q_\rho} a_{\rho,m+\sigma}\Bigg) +$$
$$+ \sum_{\tau=1}^{k-m} q_{m+\tau} \Bigg(\frac{\partial a_{s+i,\,m+\sigma}}{\partial q_{m+\tau}} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_{\rho,m+\tau} - \frac{\partial a_{s+i,\,m+\tau}}{\partial q_{m+\sigma}} +$$
$$+ \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\tau}}{\partial q_\rho} a_{\rho,m+\sigma}\Bigg)\Bigg] = 0$$
$$(i = 1, \ldots, m - s)$$

These equations must hold for arbitrary $\delta q_{m+\sigma}$ and $q'_{m+\tau}$. The expressions within the parentheses must therefore be equal to zero, that is

$$\frac{\partial a_{s+i,\,m+\sigma}}{\partial t} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_\rho - \frac{\partial a_{s+i}}{\partial a_{m+\sigma}} + \sum_{\rho=1}^{m} \frac{\partial a_{s+i}}{\partial q_\rho} a_{\rho,m+\sigma} = 0 \tag{4}$$

$$\frac{\partial a_{s+i,\,m+\sigma}}{\partial q_{m+\tau}} - \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\sigma}}{\partial q_\rho} a_{\rho,m+\tau} - \frac{\partial a_{s+i,\,m+\tau}}{\partial q_{m+\sigma}} + \sum_{\rho=1}^{m} \frac{\partial a_{s+i,\,m+\tau}}{\partial q_\rho} a_{\rho,m+\sigma} = 0$$

$$(\sigma = 1, \ldots, k - m;\ \tau = 1, \ldots, k - m;\ i = 1, \ldots, m - s)$$

which is exactly the equation (2.14).

Proof of Sufficiency. Carrying out the presentation in the reverse order, we pass from equation (2.14) to the equations (2.15), whereby the latter equation will hold for arbitrary δq_{m+1}, ..., δq_k, q'_{m+1}, ..., q'_k. Taking into account the equations (2.15) in connection with the identities (2.12), we can derive the relations (2.13). This concludes the proof.

We obtained the equations (2.14) at a point. However, if we assume that the transposition of the operations δ and $d(\ )/dt$ is permissible at each point of some region, then the equations (2.4) must hold identically in this region.

In accordance with the theorem just proven, all material systems which are subjected to constraints of the form (1.1) fall into $m + 1$ classes corresponding to the different realizable relations of the form

$$\delta \frac{d}{dt} q - \frac{d}{dt} \delta q = 0$$

6. In conclusion we consider an example which illustrates the last theorem. We take the system

$$dq_1 + q_2 dq_4 = 0, \qquad dq_2 + q_3 dq_5 = 0, \qquad dq_3 + q_1 dq_4 = 0 \qquad (2.16)$$

The integrals U of this system satisfy the following system of linear first order, partial differential equations

$$q_2 \frac{\partial U}{\partial q_1} + q_1 \frac{\partial U}{\partial q_3} - \frac{\partial U}{\partial q_4} = 0, \qquad q_3 \frac{\partial U}{\partial q_2} - \frac{\partial U}{\partial q_5} = 0$$

Following a well-known method of the integration of systems of linear, first order, partial differential equations, we can convince ourselves that this system of equations does not possess any solutions other than the constant one. This means that the system (2.16) does not possess an integrable combination.

On the other hand, the coefficients in the equations of the system (2.14) satisfy identically the equations (2.12) for $k = 5$, $m = 3$, $s = 2$, $i = 1$.

Therefore, the system (2.16) admits the transpositional relations

$$\frac{d}{dt} \delta q_3 = \delta \frac{d}{dt} q_3, \qquad \frac{d}{dt} \delta q_4 = \delta \frac{d}{dt} q_4, \qquad \frac{d}{dt} \delta q_5 = \delta \frac{d}{dt} q_5$$

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